Lecture Notes, January 15, 2009

The Arrow-Debreu Model of General Competitive Equilibrium

3.1 The Market, Commodities and Prices

N commodities

$x = (x_1, x_2, x_3, ..., x_N) \in \mathbf{R}^N$, a commodity bundle

The market takes place at a single instant, prior to the rest of economic activity.
commodity = good or service completely specified
description
location
date (of delivery)

A futures market: no reopening of trade.

Price system : $p = (p_1, p_2, ..., p_N) \neq 0$.
$p_i \geq 0$ for all i = 1, ..., N.
Value of a bundle $x \in R^N$ at prices p is $p \cdot x$.

4.1 Firms and Production Technology

F , $j \in F$, j = 1, ... , #F.
Production technology: $\mathcal{Y}^j \subset R^N$. $y \in \mathcal{Y}^j$ (the script Y notation is to emphasize that $\mathcal{Y}^j$ is bounded).
Negative co-ordinates of y are inputs; positive co-ordinates are outputs.
$y \in \mathcal{Y}^j$, $y = (-2, -3, 0, 0, 1)$

This is a more general specification than a production function. The relationship is $f^j(x) \equiv \max \{ w \mid (-x, w) \in \mathcal{Y}^j \}$.

4.2 The Form of Production Technology

P.II. $0 \in \mathcal{Y}^j$.
P.III. $\mathcal{Y}^j$ is closed. (continuity)

P.VI $\mathcal{Y}^j$ is a bounded set for each $j \in F$. (We'll dispense with this evenutally)

P.III and P.VI $\Rightarrow \mathcal{Y}^j$ is compact

Compactness of $\mathcal{Y}^j$ is needed to be sure that profit maximization is well-defined, but P.VI is an ugly assumption: boundedness of a firm's attainable production possibilities should be communicated by the price system --- not by assumption. Chapter 8 of Starr's book weakens the assumption by showing that --- even when the firm's technology set is unbounded --- under weak assumptions, the set of attainable plans is bounded. Then circumscribe the unbounded technology set by a ball strictly containing the attainable plans. Apply the analysis of chaps. 4-7 to the artificially circumscribed production technology --- there will be an equilibrium (theorem 7.1) and an equilibrium is necessarily attainable, so the circumscibing ball is not a binding constraint in equilibrium. Then delete the artificial circumscribing ball; the prices and allocation remain an equilibrium. Conclusion: P.VI can be eliminated but it's a complex pain to do so.

4. 3. Strictly Convex Production Technology

P.V. For each $j \in F$, $\mathcal{Y}^j$ is strictly convex.

Convexity implies no scale economies, no indivisibilities.

$p \in R_+^N$, $p = (p_1, p_2, ..., p_N)$, $p \neq 0$.

$\tilde{S}^j(p) \equiv \{y^{*j} \mid y^{*j} \in \mathcal{Y}^j, \quad p \cdot y^{*j} \geq p \cdot y \text{ for all } y \in \mathcal{Y}^j\}$.

Theorem 4.1: Assume P.II, P.III, P.V, and P.VI. Let $p \in R_+^N, p \neq 0$. Then $\tilde{S}^j(p)$ is a well defined continuous point-valued function.

Proof:

Well defined: $\tilde{S}^j(p)$ = maximizer of a continuous real-valued function on a compact set.

Point-valued: Strict convexity of $\mathcal{Y}^j$, P.V. Point valued-ness implies that $\tilde{S}^j(p)$ is a function.

Continuity: Let $p^\nu \in R_+^N; \nu = 1, 2, \ldots; p^\nu \neq 0$, $p^\nu \to p^o \neq 0$. Show $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^o)$.

Note: this is a consequence of the Maximum Theorem (see Berge, *Topological Spaces*), but we can provide a direct proof here, by contradiction. Suppose not. Then there is a cluster point of the sequence $\tilde{S}^j(p^\nu)$, y* so that y* $\neq \tilde{S}(p^o)$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$ (why does this inequality hold? by definition of $\tilde{S}^j(p^o)$). That is there is a subsequence p^ν so that $\tilde{S}^j(p^\nu) \to y^*$. Note that $p^\nu \cdot \tilde{S}^j(p^o) \to p^o \cdot \tilde{S}^j(p^o)$. We have $p^\nu \cdot \tilde{S}^j(p^\nu) \to p^o \cdot y^*$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$. But the dot product is a continuous function of its arguments, so for ν large, $p^\nu \cdot \tilde{S}^j(p^o) > p^\nu \cdot \tilde{S}^j(p^\nu)$, a contradiction. Hence $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^o)$. Q.E.D.

Lemma 1: (homogeneity of degree 0) Assume P.II, P.III, and P.VI. Let $\lambda > 0$, $p \in R_+^N$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$.

$$\tilde{S}(p) \equiv \sum_{j \in F} \tilde{S}^j(p)$$

4.4 Attainable Production Plans

Definition: A sum of sets $\mathcal{Y}^j$ in R^N, is defined as $\mathcal{Y} = \sum_j \mathcal{Y}^j$ is the set $\{y \mid y = \sum_j y^j \text{ for some } y^j \in \mathcal{Y}^j\}$.

Aggregate technology set:

$\mathcal{Y} \equiv \sum_{j \in F} \mathcal{Y}^j$.

Initial inputs to production $r \in R^N_+$

Definition: Let $y \in \mathcal{Y}$. Then y is said to be attainable if $y + r \geq 0$.

$y \in \mathcal{Y}$ is attainable if $(y + r) \in [\mathcal{Y} + \{r\}] \cap R^N_+$.

Note that under this definition, and P.II, P.III, P.V, P.VI the attainable set of outputs is compact and convex.